UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
GCE Advanced Level

MARK SCHEME for the October/November 2011 question paper

for the guidance of teachers

9231 FURTHER MATHEMATICS

9231/13 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes must be read in conjunction with the question papers and the report on the examination.

- Cambridge will not enter into discussions or correspondence in connection with these mark schemes.

Cambridge is publishing the mark schemes for the October/November 2011 question papers for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level syllabuses and some Ordinary Level syllabuses.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qu No	Commentary	Solution	Marks	Part Marks	Total
1	Verifies result.	$\frac{1}{n^2}-\frac{1}{(n+1)^2}=\frac{n^2+2n+1-n^2}{n^2(n+1)^2}=\frac{2n+1}{n^2(n+1)^2}$ (AG)	B1	1	
	Uses difference method	$S_N = \left(\frac{1}{1^2}-\frac{1}{2^2}\right)+\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+.......+\left(\frac{1}{N^2}-\frac{1}{(N+1)^2}\right)$	M1		
	to sum.	$=1-\frac{1}{(N+1)^2}$	A1	2	
	Considers difference between sum and sum to infinity.	$S-S_N<10^{-16}\Rightarrow\frac{1}{(N+1)^2}<10^{-16}$	M1		
		$\Rightarrow(N+1)>10^8$	A1		
	Solves inequality.	$\Rightarrow$ *least* $N=10^8$	A1	3	**[6]**
2	States proposition.	$P_n:\frac{d^n}{dx^n}\left(\frac{1}{2x+3}\right)=(-1)^n\frac{n!2^n}{(2x+3)^{n+1}}$			
	Proves base case.	$\frac{d}{dx}\left(\frac{1}{2x+3}\right)=(-1)(2x+3)^{-2}\times 2$	M1		
		$=(-1)\frac{1!\times 2}{(2x+3)^2}\Rightarrow P_1$ is true.	A1		
	States inductive hypothesis.	Assume P_k is true.			
		i.e. $\frac{d^k}{dx^k}\left(\frac{1}{2x+3}\right)=(-1)^k\frac{k!2^k}{(2x+3)^{k+1}}$	B1		
	Shows $P_k\Rightarrow P_{k+1}$.	$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{2x+3}\right)=(-1)^{k+1}\frac{2(k+1)k!2^k}{(2x+3)^{k+2}}$	M1		
		$=(-1)^{k+1}\frac{(k+1)!2^{k+1}}{(2x+3)^{k+2}}$	A1		
		$\therefore P_k\Rightarrow P_{k+1}$			
	States conclusion.	Since P_1 is true and $P_k\Rightarrow P_{k+1}$, hence by the principle of mathematical induction P_n is true $\forall n\in Z^+$.	A1	6	**[6]**

Qu No	Commentary	Solution	Marks	Part Marks	Total
3	Uses	$\sum\alpha = -5 \qquad \sum\alpha\beta = -3$	B1		
	$\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$	$\sum\alpha^2 = (-5)^2 - 2\times(-3) = 31$	M1A1	3	
	Evaluates determinant.	Det $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix} = 1-(\alpha^2+\beta^2+\gamma^2)+2\alpha\beta\gamma$	M1A1		
		$\alpha\beta\gamma = -(-15) = 15$			
		$\Rightarrow 1 - 31 + 2\times 15$	M1		
	Shows it is zero.	$= 0 \Rightarrow$ matrix is singular.	A1	4	**[7]**
4	Finds first derivative.	$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-6\sin 2t}{4\cos 2t} = -\frac{3}{2}\tan 2t$	M1A1		
	Evaluates.	When $t = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{3\sqrt{3}}{2}$	A1	3	
(ii)	Finds second derivative.	$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx} = -3\sec^2 2t \times \frac{1}{4}\sec 2t$	M1A1		
		$= -\frac{3}{4}\sec^3 2t$	A1		
	Evaluates.	When $t = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} = \frac{3}{4}\times 8 = 6$	A1	4	**[7]**
	Alternatively				
(i)	Finds cartesian equation and differentiates implicitly.	$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \Rightarrow y' = -\frac{9x}{4y} = -\frac{9}{4}\times\frac{-2\sqrt{3}}{3} = \frac{3\sqrt{3}}{2}$	M1A1 A1	3	
(ii)	Differentiates again.	$\frac{1}{2} + \frac{2}{9}\left[(y')^2 + yy''\right] = 0 \Rightarrow \frac{1}{2} + \frac{3}{2} = \frac{1}{3}y'' \Rightarrow y'' = 6$	M1A2 A1	4	**[7]**

Qu No	Commentary	Solution	Marks	Part Marks	Total
5	Binomial expansion and groups.	$(z+z^{-1})^4 = (z^4+z^{-4}) + 4(z^2+z^{-2}) + 6$,	M1A1		
		where $z = (\cos\theta + i\sin\theta)$.			
	Uses de M's Thm.	$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$	M1		
	Simplifies	$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$	A1	4	
	Integrates result	$\int_0^{\frac{\pi}{4}} \cos^4 d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}\right) d\theta$	M1		
	correctly.	$= \left[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3\theta}{8}\right]_0^{\frac{\pi}{4}}$	A1		
	Evaluates.	$= \frac{1}{4} + \frac{3\pi}{32}$	A1	3	[7]
6	Forms auxiliary equation and factorises.	$\Rightarrow m = -2 \qquad m^2 + 4m + 4 = 0$ $\Rightarrow (m+2)^2 = 0$	M1		
	States CF.	CF: $Ae^{-2t} + Bte^{-2t}$	A1		
	States form of PI	PI: $x = p\sin 2t + q\cos 2t$	M1		
	and differentiates twice.	$\dot{x} = 2p\cos 2t - 2q\sin 2t$; $\ddot{x} = -4p\sin 2t - 4q\cos 2t$			
	Compares coefficients and solves.	$\Rightarrow -8q = 1$; $8p = 0$ $\Rightarrow q = -\frac{1}{8}$; $p = 0$	M1A1		
	States GS.	GS: $x = Ae^{-2t} + Bte^{-2t} - \frac{1}{8}\cos 2t$	A1	6	
	Reason.	As $t \to \infty$ $\quad e^{-2t}$ and $te^{-2t} \to 0$	B1		
	Behaviour.	Hence x oscillates. (Accept $x \approx -\frac{1}{8}\cos 2t$.)	B1	2	[8]

Qu No	Commentary	Solution	Marks	Part Marks	Total
7	Differentiates.	$\frac{d}{dt}\{t(1+t^3)^n\} = 3t^3 n(1+t^3)^{n-1} + (1+t^3)^n$	B1		
	Rearranges.	$= 3n(1+t^3-1)(1+t^3)^{n-1} + (1+t^3)^n$	M1		
		$= (3n+1)(1+t^3)^n - 3n(1+t^3)^{n-1}$ (AG)	A1	3	
	Integrates wrt t.	$(3n+1)I_n = \left[t(1+t^3)\right]_0^1 + 3nI_{n-1}$	M1		
	Obtains reduction formula.	$(3n+1)I_n = 2^n + 3nI_{n-1}$ (AG)	A1	2	
	Evaluates I_1 (or I_0) directly. Uses reduction formula.	$I_1 = \int_0^1 (1+t^3)\,\mathrm{d}t = \left[t + 0.25t^4\right]_0^1 = 1.25$	B1M1		
	Obtains I_2.	$7I_2 = 4 + 6\times 1.25 \Rightarrow I_2 = \frac{23}{14}$	A1		
	Obtains I_3.	$10I_3 = 8 + 9\times\frac{23}{14} \Rightarrow I_3 = \frac{319}{140}$ (= 2.28)	A1	4	[9]
8	Sketches graph.	Arc above initial line. Arc below initial line.	B1 B1	2	
	Uses $\frac{1}{2}\int r^2 d\theta$	$\frac{1}{2}\int(1+\sin\theta)^2\,\mathrm{d}\theta = \frac{1}{2}\int(1+2\sin\theta+\sin^2\theta)\,\mathrm{d}\theta$	M1		
	Uses double angle formula.	$= \frac{1}{2}\int\left(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$	M1		
	Integrates.	$= \frac{1}{2}\left[\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right]+c$	M1A1		
	Inserts limits.	$A_1 = \left[\frac{1}{2}\left(\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right)\right]_0^{\frac{\pi}{2}} = \frac{3\pi}{8}+1$	M1A1		
		$A_2 = \left[\frac{1}{2}\left(\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right)\right]_{-\frac{\pi}{2}}^{0} = \frac{3\pi}{8}-1$	A1		
		$n = \left(\frac{3\pi}{8}+1\right)\div\left(\frac{3\pi}{8}-1\right) = 12.2$ (1d.p.)	A1	8	[10]

Qu No	Commentary	Solution	Marks	Part Marks	Total
9	Finds normal to plane.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -3 & 4 \end{vmatrix} = \mathbf{i} - 9\mathbf{j} - 7\mathbf{k}$	M1A1		
(i)	Deduces equation.	Π:: $x - 9y - 7z =$ constant			
		Sub e.g. (1, –1, 2) $\Rightarrow$ constant = – 4			
		Π:: $x - 9y - 7z = -4$	M1A1	4	
	General point on line inserted in plane equation to find λ.	l: $x = 6 + 2\lambda \quad y = -2 + \lambda \quad z = 1 - 4\lambda$			
		Sub in $\Pi \Rightarrow 6 + 2\lambda + 18 - 9\lambda - 7 + 28\lambda = -4$	M1		
		$\Rightarrow \lambda = -1$	A1		
		Position vector of intersection is 4**i** – 3**j** + 5**k.**	A1	3	
(ii)	Distance of point from plane formula or triple scalar product method.	Either $\left\lvert \frac{6+18-7+4}{\sqrt{1+81+49}} \right\rvert$ Or $\frac{(2i+j-4k).(i-9j-7k)}{\sqrt{1+81+49}}$	M1A1		
		$= \frac{21}{\sqrt{131}}$ (=1.83)	A1	3	
(iii)	Scalar product to find complement of angle.	$(2i + j - 4k).(i - 9j - 7k) = 21$	M1		
		$= \sqrt{4+1+16}\sqrt{1+81+49}\sin\theta$	A1		
		$\Rightarrow \sin\theta = \sqrt{\frac{21}{131}} \Rightarrow \theta = 23.6^0$ or 0.412 rad.	A1	3	[13]

Qu No	Commentary	Solution	Marks	Part Marks	Total
10	Intersections with axes.	(–1,0) , (2,0) (0,–1)	B1 B1	2	
		$yx^2 + 5xy + 10y = 5x^2 - 5x - 10$			
	Rearranges as a quadratic equation.	$(y-5)x^2 + (5y+5)x + 10(y+1) = 0$			
	Uses discriminant.	For real x $b^2 - 4ac \geq 0$			
		$\Rightarrow (5y+5)^2 - 40(y-5)(y+1) \geq 0 \ldots$	M1A1		
	Solves inequality.	$\Rightarrow (y-15)(y+1) \leq 0 \Rightarrow -1 \leq y \leq 15$ (AG)	M1A1	4	
	Finds turning points.	$y = -1 \Rightarrow x = 0$ $\quad y = 15 \Rightarrow x = -4$	M1A1 A1		
		Turning points are (-4,15) and (0,–1)			
		$y = 5$.	B1		
	States asymptote.	Axes and asymptote correct	B1		
	Sketches graph.	Graph correct.	B1B1	7	**[13]**
11	**EITHER**				
	Uses formula for mean value and integrates y wrt x,	Mean value $= \dfrac{\int_0^3 y\,dx}{3-0} = \left[\dfrac{2}{3}x^{\frac{3}{2}} - \dfrac{2}{15}x^{\frac{5}{2}}\right]_0^3 \div 3$	M1M1 A1		
	to obtain result.	$= \sqrt{3}\left[\dfrac{2}{3}\times 3 - \dfrac{2}{15}\times 9\right] \div 3 = \dfrac{4\sqrt{3}}{15}$ (= 0.462)	A1	4	
	Uses $\sqrt{1+(y')^2}$	$y' = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2}\sqrt{x} \quad \Rightarrow \dfrac{ds}{dx} = \sqrt{1 + \dfrac{1}{4}\left(\dfrac{1}{x} - 2 + x\right)}$	B1		
	and obtains result.	$\Rightarrow \dfrac{ds}{dx} = \sqrt{\dfrac{1}{4}\left(\dfrac{1}{x} + 2 + x\right)} = \left(\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2}\sqrt{x}\right)$ (AG)	B1		
	Uses correct formula and integrates to find arc length.	$s = \dfrac{1}{2}\int_0^3 \left(x^{-\frac{1}{2}} + x^{\frac{1}{2}}\right)dx = \dfrac{1}{2}\left[2\sqrt{x} + \dfrac{2}{3}x^{\frac{3}{2}}\right]_0^3 = 2\sqrt{3}$ (= 3.46)	M1A1 M1A1	6	
	Uses correct formula and integrates to obtain surface area.	$S = 2\pi\int_0^3 \left(x^{\frac{1}{2}} - \dfrac{1}{3}x^{\frac{3}{2}}\right)\left(\dfrac{1}{2}x^{-\frac{1}{2}} + \dfrac{1}{2}x^{\frac{1}{2}}\right)dx$	M1		
		$= \pi\int_0^3 \left(1 + \dfrac{2}{3}x - \dfrac{1}{3}x^2\right)dx = \left[x + \dfrac{1}{3}x^2 - \dfrac{1}{9}x^3\right]_0^3 = 3\pi$ (OE)	M1A1 M1	4	**[14]**

Qu No	Commentary	Solution	Marks	Part Marks	Total
	OR				
	Forms characteristic equation.	Det(**A** – λ**I**) = 0 $\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$	M1A1		
	Solves.	$\Rightarrow \lambda = 1,2,3$	A1A1		
	Finds eigenvectors via equations or cross-products.	$\mathbf{e}_1 = -2\mathbf{j} + \mathbf{k}$, $\mathbf{e}_2 = \mathbf{i} + \mathbf{j}$, $\mathbf{e}_3 = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$	M1A1 A1	7	
	States equation of plane.	$\mathbf{r} = s\mathbf{e} + t\mathbf{f}$ $\mathbf{A}(s\mathbf{e} + t\mathbf{f}) = s\mathbf{Ae} + t\mathbf{Af} = (s\lambda)\mathbf{e} + (t\mu)\mathbf{f}$	B1 M1A1	3	
	Cross-products of eigenvectors to obtain other plane equations.	Either $\mathbf{e}_1 \times \mathbf{e}_2 = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} \Rightarrow x - y - 2z = 0$ $\mathbf{e}_1 \times \mathbf{e}_3 = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k} \Rightarrow 2x - y - 2z = 0$ $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{i} - \mathbf{j} \Rightarrow x - y = 0$	M1A1 A1 A1	4	**[14]**